WAVES ON THE SURFACE OF A LIQUID FILM FLOWING AROUND A HORIZONTAL CYLINDER

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Wave formation on the surface of a liquid film flowing around a horizontal cylinder is studied theoretically and experimentally. Steady transverse waves are formed on the surface of the film at large velocities because of centrifugal force. The theoretical solution for the growth of these waves is obtained assuming small film thickness; the results agree closely with the experimental data.

It is known [1-3] that waves of different types are formed on the surface of a liquid film flowing around a horizontal cylinder. When the velocity of the liquid on the lower generatrix of the horizontal cylinder is small, a system of standing waves is produced which is uniformly distributed with respect to wavelength. The crests of the standing waves periodically break away in the form of drops. This system of waves is associated with the Taylor instability of the film at the bottom of the cylinder.

When the velocity of the liquid increases the acceleration perpendicular to the surface of the cylinder also increases and the Taylor instability exists over practically the entire surface of the cylinder (except near the top) and a system of steady transverse waves is formed (in the form of rolls threaded on the tube). The existence of waves of this kind was first pointed out in [4]. They were further studied theoretically and experimentally in [5], where the properties of the waves were obtained approximately without taking into account the effect of friction on the surface of the tube (in the framework of the deep water model).

Kinematic waves are also formed on a liquid film draining along the curved surface of a cylinder. These waves can only develop for large diameters (when the diameter of the cylinder is 20 times larger than the capillary constant). In the present paper we consider the Taylor instability of a liquid film flowing around a horizontal cylinder with the friction of the liquid against the wall of the cylinder taken into account.

Waves on the surface of a liquid film flowing around a cylinder were studied using the apparatus shown schematically in Fig. 1. The apparatus consists of three main parts: a stilling chamber 1, a nozzle 2, and a horizontal cylinder 3. Pure water was used as the working liquid. Water from a pump flows into the chamber through a slit in the upper part of a horizontal tube 4. The internal dimensions of the stilling chamber were 238 × 45 × 200 mm. To equalize the flow into the chamber a layer of coarse synthetic wool 6 and honeycomb 5 was used. The latter was prepared from thin nickel tubes of diameter 3 mm, length 30 mm, and wall thickness 0.1 mm. Below the stilling chamber was attached a plane nozzle of length L = 238 mm. To eliminate boundary effects in the jet, asbestos cement rods were mounted flush with the ends of the surface of the nozzle.

The arrangement of the stilling chamber and nozzle made it possible to obtain plane jets with extremely small perturbations on the surface. The experimental cylinders (interchangeable) were placed at a distance of 40 mm from the nozzle. Nozzle widths of 0.85 and 1.2 mm and cylinder diameters of 15, 26, 48 mm were used. The incident velocity of the jet u_{in} (before striking the surface of the cylinder) was varied from 0.9 to 2.5 m/sec. The velocity u_0 on the surface of the film after rotation of the jet (near the top) was somewhat smaller and was determined beforehand by measuring the velocity of particles floating on the surface of the film [6]. The experimental results are shown in Fig. 2. The results using different nozzles (curve 1: nozzle width 0.85 mm; curve 2: nozzle width 1.2 mm) are approximated by the curve.

The experiments show that for Re < 300 (Re = $Q/L\nu$, where Q is the volume flow rate of the liquid, L is the nozzle length, and ν is the viscosity of the liquid) water drains from the lower generatrix of the cylinder in the form of drops and thin water threads. This type of flow is quite well-known theoretically [2, 3] and experimentally [7]. Each thread

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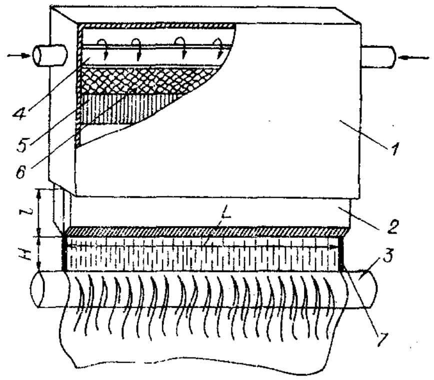

Fig. 1

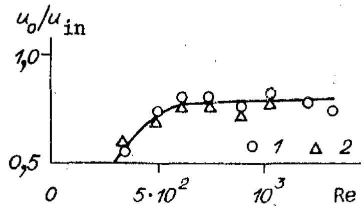

Fig. 2

corresponds to a standing transverse wave covering a small region near the bottom of the cylinder. The liquid film near the top of the cylinder remains completely smooth. When a thin knife-edge is placed under the cylinder (in the plane of the incident jet) the liquid flows over it continuously, while the water near the bottom of the cylinder flows as before.

As the velocity of the incident jet increases, the region where the surface has a noticeable wave structure extends upward from the generatrix of the cylinder. A system of regular standing rolls originates near the top of the cylinder. As the velocity increases further the wave crests begin to vibrate along the axis of the cylinder and break off from the surface without having reached the aft region. The point of separation moves upward along the surface of the cylinder as the velocity increases.

We note that as the liquid moves along the cylinder, the wave amplitude increases, while the wavelength remains constant. The wavelength decreases with increasing velocity of the incident jet and decreasing nozzle diameter. A photograph of the flow is shown in Fig. 3, where Re = 1200 and the nozzle width is 0.85 mm.

The effect of weak regular perturbations on the wave structure was studied experimentally. A system of equally spaced wires (d = 0.1 mm) was placed perpendicular to the jet at different distances from the nozzle. The spacing between the wires was varied from run to run. When the grid was placed near the nozzle the resulting perturbations on the surface of the jet were small, although their intensity was much larger than that of perturbations generated by the nozzle. The wave structure did not change as a result of such a grid. Hence one concludes that perturbations of the surface associated with the specific geometry of the nozzle also do not affect the development of waves. When the distance between the nozzle and grid was increased, the intensity of the resulting perturbations on the surface of the film also increased and the perturbations began to affect the development of the waves. Over a small interval of velocities of the incident jet the wavelength on the cylinder becomes equal to the wavelength of the perturbation (spacing between the wires) if this spacing is close to the wavelength of the unperturbed flow. Outside this interval the wavelength is somewhat smaller than that of the unperturbed flow. Most of the experiments were carried out without the use of the turbulence-generating grid. The observed wave structure was photographed and the wavelength was determined from the photographs.

We next consider the theoretical analysis of the draining of the film. A schematic diagram of the flow is shown in Fig. 4, where r, φ, z are cylindrical coordinates, v_r, v_φ, v_z are the velocities of the flow along these coordinates, λ is the wavelength, g is the acceleration due to gravity, and h is the film thickness.

The general form of the equations describing the motion of the film along the cylindrical surface are:

equations of motion

$$v_r\frac{\partial v_r}{\partial r}+\frac{v_\varphi}{r}\frac{\partial v_r}{\partial\varphi}-\frac{v_\varphi^2}{r}+v_z\frac{\partial v_r}{\partial z}=-g\cos\varphi-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}+\right.$$

$$\left.+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\varphi^2}-\frac{2}{r^2}\frac{\partial v_\varphi}{\partial\varphi}+\frac{\partial^2 v_r}{\partial z^2}\right),\quad v_r\frac{\partial v_\varphi}{\partial r}+\frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial\varphi}+\frac{v_r v_\varphi}{r}+v_z\frac{\partial v_\varphi}{\partial z}=g\sin\varphi-$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial\varphi}+\nu\left(\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial v_\varphi}{\partial r}-\frac{v_\varphi}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial\varphi^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\varphi}+\frac{\partial^2 v_\varphi}{\partial z^2}\right),$$

Fig. 3

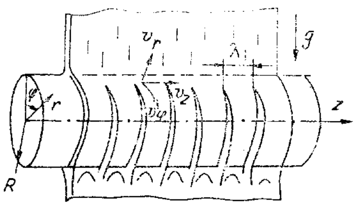

Fig. 4

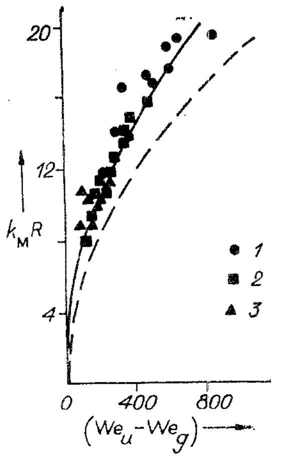

Fig. 5

$$v_r \frac{\partial v_z}{\partial r} + \frac{v_\varphi}{r}\frac{\partial v_z}{\partial \varphi} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \varphi^2} + \frac{\partial^2 v_z}{\partial z^2}\right),$$

equation of continuity

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} = 0,$$

kinematic condition on the surface of the film

$$r = R + h, \quad v_r = \frac{\partial h}{\partial \varphi}\frac{v_\varphi}{r} + \frac{\partial h}{\partial z} v_z,$$

and the boundary conditions

$$r = R, \; v_\varphi = v_r = v_z = 0.$$

We also assume that the tangential stress on the surface of the film is equal to zero (there is no friction with the air).

We write the system of equations in dimensionless form. The wavelength is taken as the unit of length in the z direction, the radius of the cylinder R is taken as the unit of length in the direction of flow along the surface of the cylinder, and the film thickness h_0 in the unperturbed state is taken as the unit of length along r. Here $h_0 = Q_L/2u_0$, where Q_L is the flow rate per unit length of the incident jet in $m^3/(sec \cdot m)$.

Waves are generated near the top of the cylinder, where the thickness of the boundary layer is small in comparison to the film thickness and the increase in v_φ because of gravity is negligible. Therefore we put $v_\varphi = u_0 \equiv \text{const}$, $g \sin\varphi = 0$, $g\cos\varphi = g$. We introduce the notation $\partial\tau = r\partial\varphi/v_\varphi$. We will also assume that the wavelength and radius are of the same order of magnitude, while the film thickness $h_0 \ll \lambda$. Then $\varepsilon = h_0/\lambda \approx h_0/R \ll 1$.

Then the system of equations has the form:

equations of motion

$$v_r^* \frac{\partial v_r^*}{\partial r^*}\varepsilon^2 + \frac{\partial v_r^*}{\partial \tau^*}\varepsilon + v_z^* \frac{\partial v_r^*}{\partial z^*}\varepsilon = \frac{1}{r^*} - \frac{gR}{v_\varphi^2} - \frac{\partial p^*}{\partial r^*} + \frac{1}{\mathrm{Re}\,\varepsilon}\left(\frac{\partial^2 v_r^*}{\partial r^{*2}}\varepsilon^3 + \frac{1}{r^*}\frac{\partial v_r^*}{\partial r^*}\varepsilon^3 - \frac{v_r^*}{r^{*2}}\varepsilon^3 + \frac{1}{r^{*2}}\frac{\partial^2 v_r^*}{\partial \varphi^{*2}}\varepsilon^3 + \frac{\partial^2 v_r^*}{\partial z^{*2}}\varepsilon^3\right),$$

$$\frac{v_r^*}{r^*}\varepsilon = \frac{1}{r^*}\frac{\partial p^*}{\partial \varphi^*} + \frac{\varepsilon}{\mathrm{Re}}\left(\frac{2}{r^{*2}}\frac{\partial v_r^*}{\partial \varphi^*}\varepsilon - \frac{1}{r^{*2}}\right), \tag{1}$$

$$v_r^* \frac{\partial v_z^*}{\partial r^*}\varepsilon + \frac{\partial v_z^*}{\partial \tau^*} + v_z^* \frac{\partial v_z^*}{\partial z^*} = -\frac{\partial p^*}{\partial z^*} + \frac{1}{\mathrm{Re}\,\varepsilon}\left(\frac{\partial^2 v_z}{\partial r^{*2}}\varepsilon^2 + \frac{1}{r^*}\frac{\partial v_z^*}{\partial r^*}\varepsilon^2 + \frac{1}{r^{*2}}\frac{\partial^2 v_z^*}{\partial \varphi^{*2}}\varepsilon^2 + \frac{\partial^2 v_z^*}{\partial z^{*2}}\varepsilon^2\right),$$

equation of continuity

$$\frac{\partial v_r^*}{\partial r^*} + \frac{v_r^*}{r^*}\varepsilon + \frac{\partial v_z^*}{\partial z^*} = 0,$$

kinematic condition on the surface of the film

$$r = R + h\colon\; v_r^* = \partial h^*/\partial\tau^* + v_z^*\partial h^*/\partial z^*,$$

and the no-slip condition

$$r = R:\ v_r^* = v_z^* = 0.$$

Here $v_z^* = v_z/v_\varphi$; $v_r^* = v_r R/v_\varphi h_0$; $z^* = z/R$; $\varphi^* = \varphi/1$; $r^* = r/R$; $\tau^* = \tau v_\varphi/R$; $h^* = h/h_0$; $p^* = p/v_\varphi^2\rho$; $\mathrm{Re} = v_\varphi h_0/\nu$. Turning to the dimensional form, we assume that the term $\mathrm{Re}\,\varepsilon$ is of order unity, ∂r^* is of order ε, and the remaining terms of the equation are of order unity. Omitting terms of order ε and also the term $v_z \partial v_z/\partial z$, since the unperturbed component $v_z = 0$, we obtain

$$\frac{\partial v_z}{\partial \tau} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\frac{\partial^2 v_z}{\partial r^2},\quad \frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{v_\varphi^2}{R} - g; \tag{2}$$

$$\partial v_r/\partial r + \partial v_z/\partial z = 0; \tag{3}$$

$$r = R + h:\ v_r = \partial h/\partial \tau + v_z \partial h/\partial z,\ r = R:\ v_r = v_z = 0; \tag{4}$$

$$r = R + h:\ \partial v_z/\partial r = 0. \tag{5}$$

The last equation follows from the vanishing of the tangential stress on the surface of the film in the long-wavelength approximation.

Hence we have transformed the original three-dimensional steady system of equations to a two-dimensional unsteady (since τ has the units of time) system. Equations similar to (2) were used in [3] to analyze the Taylor instability on the lower generatrix of the cylinder (without the term v_φ^2/r).

The pressure is found from the second equation of (2) in the form

$$\frac{1}{\rho}\int_p^{p_\sigma} dp = \int_r^{R+h} \frac{v_\varphi^2}{r}\,dr - \int_r^{R+h} g\,dr,$$
$$\frac{1}{\rho}p = \frac{1}{\rho}p_\sigma + (R + h - r)g - v_\varphi^2[\ln(R+h) - \ln r] \tag{6}$$

(p_σ is the pressure on the surface of the liquid film).

From the Young–Laplace equation in the long-wavelength approximation we have

$$p_\sigma = p_0 + \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = p_0 + \sigma\frac{1}{R+h} - \sigma\frac{\partial^2 h}{\partial z^2}, \tag{7}$$

where p_0 is the external pressure, R_1 is the radius of curvature of the cylinder, R_2 is the radius of curvature of the transverse wave, and σ is the surface tension,

Substituting (6) and (7) into the first equation of (2), we find

$$\frac{\partial v_z}{\partial \tau} = \nu\frac{\partial^2 v_z}{\partial r^2} + \frac{\sigma}{\rho}\frac{\partial^3 h}{\partial z^3} + \left[\frac{v_\varphi^2}{R+h} + \frac{\sigma}{\rho}\frac{1}{(R+h)^2} - g\right]\frac{\partial h}{\partial z}. \tag{8}$$

Because $R \gg h$, we put $R + h = R$. Integrating (8) and (3) with respect to film thickness, we obtain from (4)

$$\frac{\partial}{\partial \tau}\int_R^{R+h} v_z dr = -\nu\left(\frac{\partial v_z}{\partial r}\right)_{r=R} + \frac{\sigma h}{\rho}\frac{\partial^3 h}{\partial z^3} + \left[\frac{v_\varphi^2 h}{R} + \frac{\sigma h}{\rho R^2} - gh\right]\frac{\partial h}{\partial z},$$
$$\frac{\partial h}{\partial \tau} + \frac{\partial}{\partial z}\int_R^{R+h} v_z dr = 0. \tag{9}$$

The term $-v_z(\partial h/\partial \tau)_{r=R+h}$ is omitted on the left-hand side of the first equation of (9), since it can be shown that its ratio with $\partial/\partial\tau \int_R^{R+h} v_z dr$ is the same order as the ratio of the inertial terms omitted in (2) with $\partial v_z/\partial \tau$.

The velocity profile v_z must be known in order to determine the quantity $(\partial v_z/\partial r)_{r=R}$. We estimate v_z from the first equation of (2), assuming a pressure gradient $\partial p/\partial z$ and $\partial v_z/\partial \tau = 0$. Integrating this equation with the use of (5), we obtain the semiparabolic velocity distribution

$$v_z = \frac{1}{2\rho\nu(R+h)^2}\frac{\partial p}{\partial z}\left[\frac{2(r-R)}{R+h} - \frac{(r-R)^2}{(R+h)^2}\right] = U_z(z)[2\eta - \eta^2]. \tag{10}$$

This assumption was justified in [8]. We next introduce the instantaneous flow rate

$$q=\int_{R}^{R+h} v_z dr$$

and from (9) and (10) we obtain

$$\frac{\partial q}{\partial \tau}=-\frac{3\nu}{h^2}q-\frac{\sigma h}{\rho}\frac{\partial^3 h}{\partial z^3}+\left[\frac{v_\varphi^2 h}{R}+\frac{\sigma h}{\rho R^2}-gh\right]\frac{\partial h}{\partial z},\quad \frac{\partial q}{\partial z}+\frac{\partial h}{\partial \tau}=0. \tag{11}$$

Substituting $q = q_0 + q'$ and $h = h_0 + h'$ into (11), where a prime denotes the perturbed part, using the fact that $q_0 = 0$, $h_0 \gg h'$, and keeping terms of order h_0, we have

$$\frac{\partial q'}{\partial \tau}=-\frac{3\nu}{h_0^2}q'+\frac{\sigma h_0}{\rho}\frac{\partial^3 h'}{\partial z^3}+\left[\frac{v_\varphi^2 h_0}{R}+\frac{\sigma h_0}{\rho R^2}-gh_0\right]\frac{\partial h'}{\partial z}; \tag{12}$$

$$\partial q'/\partial z+\partial h'/\partial \tau=0. \tag{13}$$

Differentiating (13) with respect to τ, we can find $\partial q'/\partial z$ and $\partial^2 q'/\partial z\partial\tau$ from the resulting equation and from (13). Differentiating (12) with respect to z and substituting the derivatives of the instantaneous flow rate in the resulting expression, we obtain the following equation for the film thickness perturbation:

$$\frac{\partial^2 h'}{\partial \tau^2}+\frac{3\nu}{h_0^2}\frac{\partial h'}{\partial \tau}+\frac{\sigma h_0}{\rho}\frac{\partial^4 h'}{\partial z^4}+\left[\frac{v_\varphi^2 h_0}{R}+\frac{\sigma h_0}{\rho R^2}-gh_0\right]\frac{\partial^2 h'}{\partial z^2}=0. \tag{14}$$

We assume a solution to (14) in the form of standing waves in z which are amplified (damped) in τ:

$$h'=A_0\exp(ikz)\exp\beta\tau \tag{15}$$

(k is the wave number and β is the growth constant of the wave). Substituting (15) into (14), we obtain the dispersion relation

$$\beta^2+\frac{3\nu}{h_0^2}\beta+\frac{\sigma h_0}{\rho}k^4-\left[\frac{v_\varphi^2 h_0}{R}-\frac{\sigma h_0}{\rho R^2}-gh_0\right]k^2=0, \tag{16}$$

and hence the growth constant of the wave

$$\beta=-\frac{3\nu}{2h_0^2}\pm\sqrt{\left(\frac{3\nu}{2h_0^2}\right)^2-\frac{\sigma h_0}{\rho}k^4+\left[\frac{v_\varphi^2 h_0}{R}+\frac{\sigma h_0}{\rho R^2}-gh_0\right]k^2}. \tag{17}$$

For waves corresponding to maximum growth

$$\frac{d\beta}{dk}=\frac{-4\frac{\sigma h_0}{\rho}k^3+2\left[\frac{v_\varphi^2 h_0}{R}+\frac{\sigma h_0}{\rho R^2}-gh_0\right]k}{\sqrt{\left(\frac{3\nu}{2h_0^2}\right)^2-\frac{\sigma h_0}{\rho}k^4+h_0\left[\frac{v_\varphi^2}{R}+\frac{\sigma}{\rho R^2}-g\right]k^3}}. \tag{18}$$

Putting $\beta = 0$, we obtain from (16) the wave number corresponding to neutral waves

$$k_0=\sqrt{\frac{\rho}{\sigma h_0}\left[\frac{v_\varphi^2 h_0}{R}+\frac{\sigma h_0}{\rho R^2}-gh_0\right]}.$$

For waves corresponding to maximum growth we obtain from (18)

$$k_M=\sqrt{\frac{\rho}{2\sigma h_0}\left[\frac{v_\varphi^2 h_0}{R}+\frac{\sigma h_0}{\rho R^2}-gh_0\right]}, \tag{19}$$

and in dimensionless form

$$k_M^*=k_M R=\frac{1}{\sqrt{2}}\sqrt{\mathrm{We}_u-\mathrm{We}_g+1}$$

$$(\mathrm{We}_u=v_\varphi^2 R\rho/\sigma,\ \mathrm{We}_g=gR^2\rho/\sigma).$$

Substituting (19) into (17), we find the increment for waves corresponding to maximum growth

$$\beta_M=-\frac{3\nu}{2h_0^2}+\sqrt{\left(\frac{3\nu}{2h_0^2}\right)^2+\frac{\rho h_0}{4\sigma}\left[\frac{v_\varphi^2}{R}+\frac{\sigma}{\rho R^2}-g\right]^2}.$$

The spatial growth increment is found by substituting $\tau = \varphi r/v_\varphi$ into (15). Then $\beta = \beta_M R/v_\varphi$.

It is interesting to note that from (17) perturbations of any wavelength moving along the axis of the cylinder (β has an imaginary part) are damped (the real part of β is negative).

The experimental and calculated results are compared in Fig. 5 in the form of the dependence $k_M R = f(We_u - We_g)$. Satisfactory agreement is observed between the calculations (solid curve) and experiment. The dashed curve shows the calculated results using the formula of [5], which was obtained without taking into account the effect of the cylinder wall on the stability of the flow. Experimental results were obtained for cylinders with d = 48, 26, and 15 mm (points 1-3).

It is evident from the photograph of the flow (Fig. 3) that the distance between rolls varies somewhat along the length of the cylinder. Therefore the average value of the wavelength was found for each flow regime.

Our results determine the nature of the waves produced on the surface of a film flowing rapidly along the surface of a cylinder. The cause of wave formation is the Taylor instability of the film surface induced by the centrifugal force.

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